NTUA Summer School, June 2018. Introduction to dispersive PDE.

Exercises

Problem 1. (a) Let $\phi \in \mathbb{S}(\mathbb{R})$ and $z \in \mathbb{C}\backslash\{0\}$ with a nonnegative real part. Then

$$\int_{\mathbb{R}} e^{-z|x|^2} \widehat{\phi}(x) dx = \frac{1}{\sqrt{2z}} \int_{\mathbb{R}} e^{-\frac{|x|^2}{4z}} \phi(x) dx.$$

(b) Assuming that $g \in \mathbb{S}(\mathbb{R})$, express the solution of

$$\begin{cases} iu_t + u_{xx} = 0, x \in \mathbb{R}, t \in \mathbb{R}, \\ u(0, \cdot) = g(\cdot) \in H^s(\mathbb{R}) \end{cases} \tag{1}$$

as a convolution of the tempered distribution $\frac{1}{\sqrt{4\pi i t}} e^{i\frac{|x|^2}{4t}}$ with g.
(c) Similarly, prove that the solution of the linear Schrödinger equation on $\mathbb{R}^n$

$$\begin{cases} iu_t + \Delta u = 0, x \in \mathbb{R}^n, t \in \mathbb{R}, \\ u(0, \cdot) = g(\cdot) \in \mathbb{S}(\mathbb{R}^n) \end{cases} \tag{2}$$

is given by

$$e^{it\Delta} g = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} g(y)\, dy.$$

(d) Conclude that the following dispersive estimate holds

$$\left\| e^{it\Delta} g \right\|_{L^\infty_x(\mathbb{R}^n)} \le \frac{1}{(4\pi |t|)^{\frac{n}{2}}} \|g\|_{L^1(\mathbb{R}^n)}.$$

Problem 2. Consider the initial value problem

$$\begin{cases} iu_t + \Delta u = |u|^{p-1} u, \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad \text{p } an\ odd\ integer, \\ u(x, 0) = u_0(x) \in \mathcal{S}(\mathbb{R}^n). \end{cases} \tag{3}$$

Show that smooth solutions of the above equation satisfy the following conservation laws

$$\|u(t)\|_{L^2_x} = \|u_0\|_{L^2},$$

$$E(u)(t) = \frac{1}{2} \int |\nabla u(t)|^2 dx + \frac{1}{p+1} \int |u(t)|^{p+1} dx = E(u_0),$$

$$\vec{p}(t) = \Im \int_{\mathbb{R}^n} \overline{u}\ \nabla u dx = \vec{p}(0).$$

Problem 3. Duhamel's principle. Let I be any time interval and suppose that $u \in C_t^1 \mathcal{S}(I \times \mathbb{R}^n)$ and that $F \in C_t^0 \mathcal{S}(I \times \mathbb{R}^n)$. Then u solves

$$\begin{cases} iu_t + \Delta u = F, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,t_0) = u(t_0) \in \mathcal{S}(\mathbb{R}^n), \end{cases} \tag{4}$$

if and only if

$$u(x,t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-s)\Delta} F(x,s) ds.$$

Problem 4. Fix $n \geq 1$. Consider the solution of

$$\begin{cases} iu_t + \Delta u = F, x \in \mathbb{R}^n, t \in \mathbb{R}, \\ u(0,\cdot) = g(\cdot) \in \mathbb{S}(\mathbb{R}^n). \end{cases} \tag{5}$$

Using Problem 3 we can express the solution as

$$u(x,t) = U(t)g(x) - i \int_0^t U(t-s)F(x,s)ds,$$

where $U(t)g$ is the linear evolution. We call a pair (q,r) of exponents admissible if $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $(q,r,n) \neq (2,\infty,2)$. Prove that for any admissible exponents (q,r) and $(\tilde{q},\tilde{r})$ we have the following estimates: The linear estimate

$$\|U(t)u_0\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L^2}, \tag{6}$$

and the nonlinear estimate

$$\| \int_0^t U(t-s)F(x,s)ds \|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^n)} \tag{7}$$

where $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ and $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$.

Problem 5 (Gronwall's inequality) Assume that for a.e. $t \in [0,T]$, we have

$$f(t) \leq A + \int_0^t g(\tau) f(\tau) d\tau$$

for some $A \geq 0$ and some nonnegative functions f and g such that $fg \in L^1([0,T])$. Prove that

$$f(t) \leq A \exp\left(\int_0^t g(\tau) d\tau \right), \quad t \in [0,T].$$

Problem 6. Recall Einstein's summation convention for summing tensors. Let $\nabla_k = \frac{\partial}{\partial x_k}$. If u is a smooth solution of (3) and $\rho = |u|^2$ and $p_k = \Im(\overline{u}\nabla_k u)$, then the following local conservation laws are true:

$$\partial_t \rho + 2\nabla_j p^j = 0,$$

$$\partial_t p^j + \nabla^k \Big(\delta_k^j \Big(-\frac{1}{2}\Delta\rho + \frac{p-1}{p+1}|u|^{p+1} \Big) + \sigma_k^j \Big) = 0$$

where the symmetric tensor $\sigma_{jk} = 2\Re(\nabla_j u \nabla_k \overline{u})$.

Notice that if we integrate the first quantity we obtain the conservation law of mass while integration of the second quantity leads to momentum conservation.

Problem 7 a) Prove the Hardy-Littlewood-Sobolev inequality in all dimensions

$$\|f \star (|y|^{-\gamma})\|_{L^q(\mathbb{R}^n)} \leq C_{p,q}\|f\|_{L^p(\mathbb{R}^n)}$$

for $0 < \gamma < n$, $1 < p < q < \infty$, and $\frac{1}{q} = \frac{1}{p} - \frac{n-\gamma}{n}$.

Hint: Review the basic properties of the maximal function

$$M(f)(x) = \sup_{\delta > 0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)|dy.$$

b) Derive Young's inequality by the Riesz-Thorin interpolation theorem.

c) Prove the distributional identity for $0 < \alpha < n$,

$$\widehat{(|x|^{-\alpha})}(\xi) = c_{\alpha,n}|\xi|^{\alpha-n}$$

where $c_{\alpha,n}$ is a constant depending only on α and n.

d) Use parts a), c) and duality to prove the Sobolev embedding

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C\||\nabla|^s f\|_{L^2(\mathbb{R}^n)}$$

with $\frac{1}{2} = \frac{1}{p} + \frac{s}{n}$ and $2 < p < \infty$.

e) Use part d) to prove the following Gagliardo-Nirenberg inequality on $\mathbb{R}^n$, for $\frac{1}{p} = \frac{1}{2} - \frac{\theta}{n}$,

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^2}^{\theta}\|u\|_{L^2}^{1-\theta}.$$

Problem 8 a) Let $f \in C_0^\infty([a,b])$ and $\phi'(x) \neq 0$ for any $x \in [a,b]$. Then

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)} f(x)dx = O(\lambda^{-k}), \text{ as } \lambda \to \infty$$

for any $k \in \mathbb{Z}^+$.

b) Let $k \in \mathbb{Z}^+$ and $|\phi^{(k)}(x)| \geq 1$ for any $x \in [a,b]$ with $\phi'(x)$ monotonic when $k = 1$. Then

$$\Big| \int_a^b e^{i\lambda\phi(x)} dx \Big| \leq c_k \lambda^{-\frac{1}{k}}$$

where the constant c_k is independent of a and b.

c) Van der Corput Lemma. Under the hypothesis of part b) prove that

$$\Big| \int_a^b e^{i\lambda\phi(x)} f(x) dx \Big| \leq c_k \lambda^{-\frac{1}{k}} \Big(\|f\|_{L^\infty} + \|f'\|_{L^1} \Big).$$

Problem 9 Distributional solutions of NLS.

a) Show that $e^{it\Delta}u_0 \in C_t^0 L_x^2(\mathbb{R} \times \mathbb{R}^n)$ if $\|u_0\|_{L^2(\mathbb{R}^n)} < \infty$.

b) Show that if $u_0 \in L^2$, then for any $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n)$ we have

$$\int\int_{\mathbb{R}\times\mathbb{R}^n} L'(\phi(x,t))\ e^{it\Delta} u_0(x)\ dxdt = 0, \tag{8}$$

where L' is the formal adjoint of the operator $L = i\partial_t + \Delta$. Thus we say that $e^{it\Delta}u_0$ satisfies (1) in the sense of distributions.

c) Assume $F \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$. Then we know that $H(F)(x,t) = -i\int_0^t e^{i(t-s)\Delta}F(x,s)ds$ is a C^∞ funtion that satisfies (4) with zero initial data. First show that

$$\|H(F)(t)\|_{L_x^2} \leq |t|^{\frac{1}{2}} \|F\|_{L_t^2 L_x^2}.$$

Then show that $H(F) \in C_t^0 L_x^2(\mathbb{R} \times \mathbb{R}^n)$.

d) Use Strichartz estimates and prove that $H(F) \in C_t^0 L_x^2(\mathbb{R} \times \mathbb{R}^n)$ when $F \in L_t^{q'} L_x^{r'}$ for all (q,r) Strichartz admissible exponents and (q', r') their Hölder dual.

e) Now consider the solution to the L^2 sub-critical problem that we solved in class. Thus assume that for $u_0 \in L^2$, u solves

$$u(x,t) = e^{i(t)\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}|u|^{p-1}u(s)ds$$

in $C_t^0 L_x^2 \cap L_t^q L_x^r$ with $p < 1 + \frac{4}{n}$. Then show that u solves $iu_t + \Delta u = |u|^{p-1}u$ in the sense of distributions.

Problem 10. For $n = 1$ prove that

$$\sup_x \int_{-\infty}^{\infty} |D_x^{\frac{1}{2}} e^{it\partial_{xx}} u_0|^2\ dt \leq C\|u_0\|_{L^2(\mathbb{R})}^2.$$

Problem 11. For $s \geq 0$, prove that

$$\sup_x \left\| \eta(t) e^{it\partial_{xx}} g \right\|_{H_t^{\frac{2s+1}{4}}} \lesssim \|g\|_{H^s},$$

where $\eta \in C_0^\infty(\mathbb{R})$.

Problem 12 Prove that the number of divisors, $d(N)$, of an integer N is bounded by $C_\epsilon N^\epsilon$ for any $\epsilon > 0$ by following the steps below:

(a) Prove that the number of divisors of $N = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ is equal to

$$\prod_{j=1}^{k} (a_j + 1).$$

(b) Prove that $\frac{a+1}{p^{\epsilon a}}$ is bounded in $a \in \mathbb{N}$ by a constant depending on ϵ for any prime p.

(c) Prove that $\frac{a+1}{p^{\epsilon a}}$ is bounded by 1 if $p^\epsilon > e$.

(d) Complete the proof by noting that the number of primes less than $e^{1/\epsilon}$ contributing to the product $\frac{d(N)}{N^\epsilon}$ is uniformly bounded in N.

Problem 13. Define the operators $\Gamma_j = x_j + 2it\partial_{x_j}$, $j = 1, ..., n$.

i) Prove that for any multiindex α

$$\Gamma^\alpha u(x,t) = e^{\frac{i|x|^2}{4t}} (2it\partial_x)^\alpha e^{-\frac{i|x|^2}{4t}} u(x,t) = e^{it\Delta} x^\alpha e^{-it\Delta} u(x,t).$$

ii) Prove that Γ_j commutes with $\partial_t - i\Delta$.

iii) If $u_0 \in L^2(\mathbb{R}^n)$ and $x^\alpha u_0 \in L^2(\mathbb{R}^n)$, show that $\Gamma^\alpha u \in C(\mathbb{R} : L^2(\mathbb{R}^n))$ and so

$$\partial_x^\alpha \left(e^{-\frac{i|x|^2}{4t}} e^{it\Delta} u_0 \right) \in C(\mathbb{R} \setminus \{0\} : L^2(\mathbb{R}^n).)$$

iv) If $u_0 \in H^s(\mathbb{R}^n)$, $s \in \mathbb{Z}^+$ and $x^\alpha u_0 \in L^2(\mathbb{R}^n)$, $|\alpha| \leq s$, then

$$u = e^{it\Delta} u_0 \in C\Big(\mathbb{R} : H^s \cap L^2(|x|^s dx)\Big).$$

Problem 14 Prove that there do not exist p, q, t with $1 \leq q < p < \infty$, $t \in \mathbb{R} \setminus \{0\}$ such that

$$e^{it\Delta} : L^p(\mathbb{R}^n) \mapsto L^q(\mathbb{R}^n) \text{ is continuous.}$$

i) Prove that $e^{it\Delta}$ commutes with translations. That is if $\tau_h f(x) = f(x - h)$ then

$$\tau_h(e^{it\Delta} f(x)) = e^{it\Delta} \tau_h f(x).$$

ii) Prove that if $f \in L^p(\mathbb{R}^n)$ then

$$\lim_{|h|\to\infty} \|f + \tau_h f\|_{L^p(\mathbb{R}^n)} = 2^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)}.$$

iii) Using ii) prove that $\|e^{it\Delta} f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ implies that

$$\|e^{it\Delta} f\|_{L^q(\mathbb{R}^n)} \leq C 2^{(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)}$$

which leads to a contradiction.

<u>Problem 15</u> Assume $p < 1 + \frac{4}{n-2}$ for $n \geq 3$ ($1 < p < \infty$ for $n = 1, 2$) and $a > 0$ and $b \in \mathbb{R}$. If $u \in H^1(\mathbb{R}^n)$ satisfies

$$-\Delta u + au = b|u|^{p-1}u \in H^{-1}(\mathbb{R}^n)$$

then the following properties hold:

$$\text{i)} \int_{\mathbb{R}^n} |\nabla u|^2 dx + a \int_{\mathbb{R}^n} |u|^2 dx = b \int_{\mathbb{R}^n} |u|^{2+\frac{4}{n}} dx.$$

$$\text{ii)(Pohozaev's identity)} \quad (n-2) \int_{\mathbb{R}^n} |\nabla u|^2 dx + an \int_{\mathbb{R}^n} |u|^2 dx = \frac{2bn}{2+\frac{4}{n}} \int_{\mathbb{R}^n} |u|^{2+\frac{4}{n}} dx.$$

Now consider the energy functional $E(u)(t)$ for the equation $iu_t + \Delta u + |u|^{\frac{4}{n}} u = 0$. Show that $E(Q) = 0$, where Q is the ground state. Recall that Q is the unique, symmetric, positive solution of the elliptic equation $-\Delta u + u = |u|^{\frac{4}{n}} u$ in $\mathbb{R}^n$.

<u>Problem 16</u> For any $b > \frac{1}{2}$ show that $X^{s,b}$ embedds into $C(\mathbb{R} : H^s(\mathbb{R}))$.

<u>Problem 17</u> a) For any $b > 1/2$ prove that

$$\int_{\mathbb{R}} \frac{1}{\langle x \rangle^{2b} \sqrt{|x-\beta|}} dx \lesssim \frac{1}{\langle \beta \rangle^{1/2}},$$

$$\int_{\mathbb{R}} \frac{1}{\langle x-\alpha \rangle^{2b} \langle x-\beta \rangle^{2b}} dx \lesssim \frac{1}{\langle \alpha-\beta \rangle^{2b}}.$$

b) For $\beta \in (0, 1]$, we have

$$\int_{\mathbb{R}} \frac{d\tau}{\langle \tau+\rho_1 \rangle^{\beta} \langle \tau+\rho_2 \rangle} \lesssim \frac{1}{\langle \rho_1-\rho_2 \rangle^{\beta-}}.$$

<u>Problem 18</u> Consider the initial value problem with periodic boundary conditions

$$\begin{cases} iu_t + u_{xx} \pm |u|^{p-1}u = 0, & x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ u(0, \cdot) = g(\cdot) \in H^s(\mathbb{T}), \end{cases} \tag{9}$$

for $s \geq 0$. We know that solutions of (9) conserve the L^2-norm. Prove that if the local existence time δ in H^s, depends only on the L^2 norm of the initial data, then the following global bound holds

$$\|u(t)\|_{H^s} \leq Ce^{C|t|}\|u(0)\|_{H^s}.$$

<u>Problem 19</u> Use the algebra property of $H^s(\mathbb{T})$, $s > \frac{1}{2}$ and Duhamel's principle to prove that (9) is locally wellposed in $C^0_t H^s_x$.

<u>Problem 20</u> Prove, using the Gagliardo–Nirenberg inequality, that the smooth solutions of (9) satisfy

$$\|u\|_{H^1(\mathbb{T})} \leq C = C(\|g\|_{H^1}),$$

both in the focusing and defocusing cases.

<u>Problem 21</u> For $s \geq 1$, prove that

$$\||u|^2 u\|_{H^s(\mathbb{T})} \lesssim \|u\|^2_{H^1(\mathbb{T})}\|u\|_{H^s(\mathbb{T})}.$$

<u>Problem 22</u> Prove using the previous two exercises, and Gronwall's inequality prove that the smooth solutions of the cubic NLS equation (9) satisfy for $s \geq 1$

$$\|u\|_{H^s(\mathbb{T})} \leq \|g\|_{H^s(\mathbb{T})}e^{Ct}.$$

Conclude that (9) is globally wellposed on $H^s(\mathbb{T})$, $s \geq 1$.

<u>Problem 23</u> a) Prove the following If $\beta \geq \gamma \geq 0$ and $\beta + \gamma > 1$, then

$$\sum_n \frac{1}{\langle n - k_1\rangle^\beta \langle n - k_2\rangle^\gamma} \lesssim \langle k_1 - k_2\rangle^{-\gamma}\phi_\beta(k_1 - k_2)$$

where

$$\phi_\beta(k) := \sum_{|n|\leq|k|} \frac{1}{|n|^\beta} \sim \begin{cases} 1, & \beta > 1, \\ \log(1 + \langle k\rangle), & \beta = 1, \\ \langle k\rangle^{1-\beta}, & \beta < 1. \end{cases}$$

b) If $\beta > 1/2$, then

$$\sum_n \frac{1}{\langle n^2 + c_1 n + c_2\rangle^\beta} \lesssim 1,$$

where the implicit constant is independent of c_1 and c_2.

<u>Problem 24</u> In this exercise we describe how one obtains the well-posedness of the BBM equation

$$\begin{cases} u_t - u_{txx} + u_x + uu_x = 0, \;\; x \in \mathbb{R}, \; t \in \mathbb{R}, \\ u(0, x) = g(x) \in H^s(\mathbb{R}), \;\; s \geq 0. \end{cases} \tag{10}$$

(a) Show that the smooth solutions satisfy the conservation law

$$E(u(t)) := \int_{\mathbb{R}} u^2 dx + \int_{\mathbb{R}} u_x^2 dx = E(g).$$

(b) Show that for any $s \geq 0$

$$\left\| \frac{\partial_x}{1-\partial_{xx}} \left(u^2\right) \right\|_{H^s} \lesssim \|u\|_{H^s}^2.$$

(c) Obtain local well-posedness in $C_t^0 H_x^s$ for any $s \geq 0$ with the local existence time depending on $\|g\|_{H^s}$.
(d) Obtain global well-posedness in H^s, $s \geq 1$. In fact, global well-posedness holds in L^2 by a variation of the high–low decomposition method of Bourgain. This result is optimal.

<u>Problem 25</u> Consider the defocusing NLS

$$\begin{cases} iu_t + \Delta u = |u|^{p-1}u \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^n) \end{cases} \tag{11}$$

for any $1 < p < 1 + \frac{4}{n-2}$, $n \geq 3$ ($1 < p < \infty$ for $n = 1, 2$). If in addition $\|xu_0\|_{L^2} < \infty$ and

$$u \in C_t^0(\mathbb{R}; H^1(\mathbb{R}^n))$$

solves (11), then we have: If $p > 1 + \frac{4}{n}$ then for any $2 \leq r \leq \frac{2n}{n-2}$ ($2 \leq r \leq \infty$ if $n = 1$, $2 \leq r < \infty$ if $n = 2$)

$$\|u(t)\|_{L^r} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{r})}$$

for all $t \in \mathbb{R}^n$.

<u>Problem 26</u> Consider the first Picard iteration for the KdV

$$e^{-t\partial_x^3} \int_0^t e^{t'\partial_x^3} \left[e^{-t'\partial_x^3} g \partial_x \left(e^{-t'\partial_x^3} g\right)\right] dt'.$$

Show that on the Fourier side (ignoring zero modes) the term can be written as

$$\sum_{k_1+k_2=k} \frac{\widehat{g}(k_1)\widehat{g}(k_2)}{-3ikk_1} (e^{-3ik_1k_2kt} - 1).$$

Show that this term is in $H^1(\mathbb{T})$ if $g \in L^2(\mathbb{T})$.

<u>Problem 27</u> Show that if

$$\sup_{\xi,\tau} \left(\int_{\xi_1,\xi_2} \int_{\tau_1,\tau_2} M^2 \right) < \infty$$

where

$$M(\xi_1, \xi_2, \xi, \tau_1, \tau_2, \tau) = \frac{\langle \xi \rangle^{s+a} \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} \langle \xi - \xi_1 + \xi_2 \rangle^{-s}}{\langle \tau - \xi^2 \rangle^{1-b} \langle \tau_1 - \xi_1^2 \rangle^b \langle \tau_2 - \xi_2^2 \rangle^b \langle \tau - \tau_1 + \tau_2 - (\xi - \xi_1 + \xi_2)^2 \rangle^b}$$

then the following inequality is true

$$\| |u|^2 u \|_{X^{s+a,b-1}(\mathbb{R})} \lesssim \|u\|^3_{X^{s,b}(\mathbb{R})}.$$

Problem 28 Consider the KdV equation in the form

$$u_t + uu_x + u_{xxx} = 0. \tag{12}$$

A *travelling wave solution* is a function

$$u(x, t) = f(x - ct)$$

that satisfies (12). The two basic features of any travelling wave are the underlying profile shape defined by f and the speed $|c|$ at which the profile is translated along the $x-$ axis. It is assumed that f is not constant and c is not zero in order for $u(x, t)$ to represent the movement of a disturbance through a medium. Follow the steps below to obtain a travelling wave solution for the KdV equation.

1. Substitute $u(x, t) = f(x - ct)$ into (12) to obtain a third order equation (ODE) for f. Assume that $f(z)$ satisfies $f, f', f'' \to 0$ as $z \to \infty$.
2. Integrate once the equation you obtained in the previous step.
3. Multiply the resulting equation with f' and integrate the result.
4. Set $g^2 = 3c - f$, assuming $0 < f < 3c$ and integrate the new equation which is a first order equation for g.

Your final answer should be

$$u(x, t) = 3c\ sech^2 \left[\frac{\sqrt{c}}{2}(x - ct) \right].$$

The solution is a *soliton*. It is a pulse that travels at constant speed while maintaining its shape.